MIT 6.875 & Berkeley CS276

Foundations of Cryptography
Lecture 10

Today:
Constructions of Public-Key Encryption

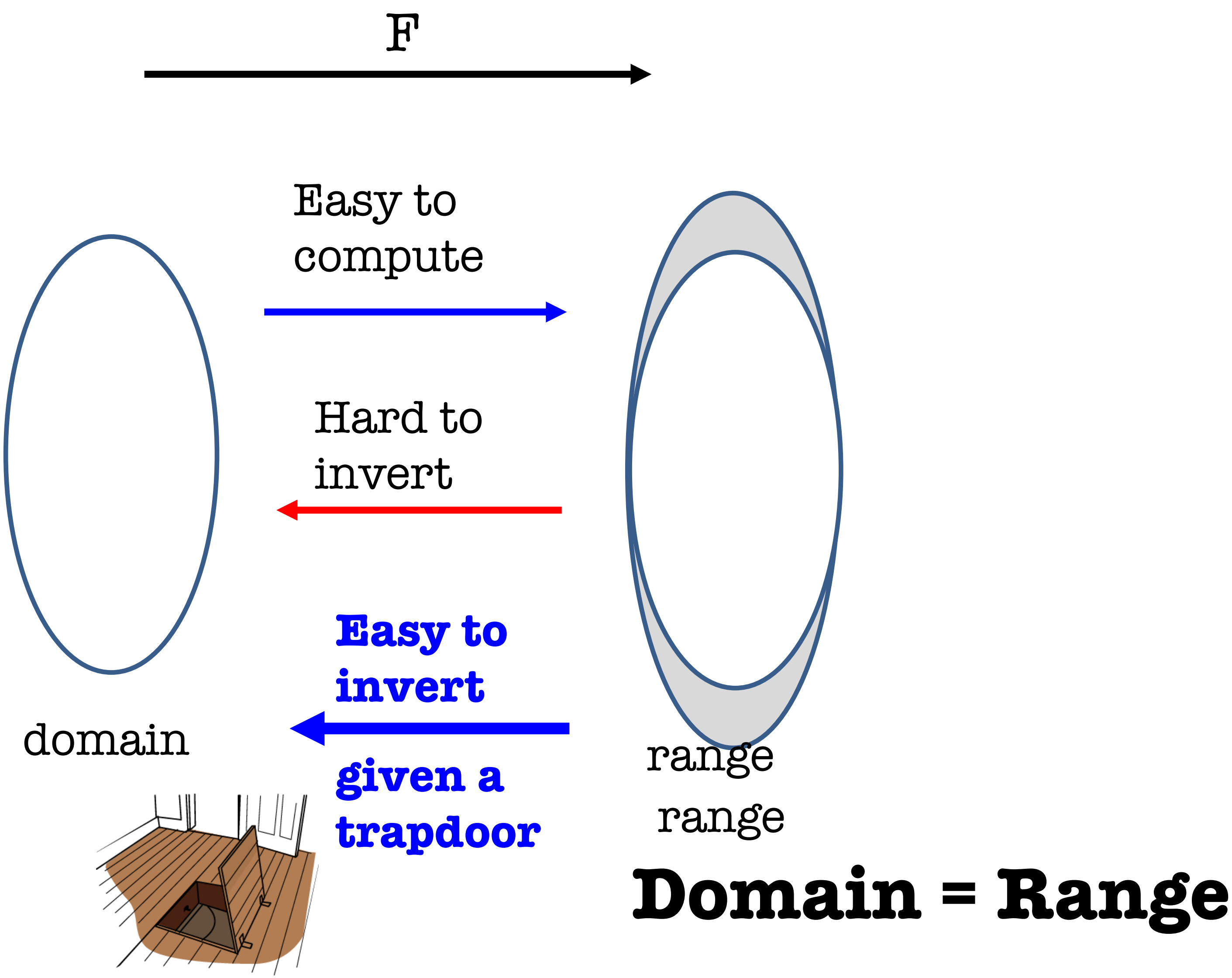
1: Trapdoor Permutations (RSA) *composite N/factoring*

2: Quadratic Residuosity/Goldwasser-Micali *composite N/factoring*

3: Diffie-Hellman/El Gamal *prime p/discrete log*

4: Learning with Errors/Regev *small numbers, large dimensions*

Trapdoor One-way Functions
Trapdoor Permutations



Review: Number Theory

Let's review some number theory from L7-8.

Let $N = pq$ be a product of two large primes.

Fact: $Z_N^* = \{a \in Z_N : \gcd(\mathrm{a}, \mathrm{N}) = 1\}$ is a group.

- group operation is multiplication mod N.
- inverses exist and are easy to compute (how so?)
- the order of the group is $\phi(N) = (p-1)(q-1)$

Lecture 8: The map $F(x) = x^2 \bmod N$ is a 4-to-1 trapdoor function, as hard to invert as factoring N.

The RSA Trapdoor Permutation

Today: Let e be an integer with $\gcd(e, \phi(N)) = 1$. Then, the map $F_{N,e}(x) = x^e \bmod N$ is a trapdoor permutation.

Key Fact: Given d such that $ed = 1 \bmod \phi(N)$, it is easy to compute x given x^e.

Proof: $(x^e)^d$

This gives us the RSA trapdoor permutation collection.

$$\{F_{N,e} : \gcd(e, N) = 1\}$$

Trapdoor for inversion: $d = e^{-1} \bmod \phi(N)$.

The RSA Trapdoor Permutation

Today: Let e be an integer with $\gcd(e, \phi(N)) = 1$. Then, the map $F_{N,e}(x) = x^e \bmod N$ is a trapdoor permutation.

Hardness of inversion without trapdoor = **RSA assumption**

given N, e (as above) and x^e mod N, hard to compute x.

We know that if factoring is easy, RSA is broken (and that's the only *known* way to break RSA)

Major Open Problem: Are factoring and RSA equivalent?

The RSA Trapdoor Permutation

Today: Let e be an integer with $\gcd(e, \phi(N)) = 1$. Then, the map $F_{N,e}(x) = x^e \bmod N$ is a trapdoor permutation.

Hardcore bits (galore) for the RSA trapdoor one-way perm:

- The Goldreich-Levin bit $\mathrm{GL}(r; r') = \langle r, r' \rangle \bmod 2$
- The least significant bit $\mathrm{LSB}(r)$
- The "most significant bit" $HALF_N(r) = 1$ iff $r < N/2$
- In fact, any single bit of r is hardcore.

RSA Encryption

- $Gen(1^n)$: Let $N = pq$ and (e, d) be such that $ed = 1 \; mod \; \phi(N)$.

  Let $pk = (N, e)$ and let $sk = d$.

- $Enc(pk, b)$ where b is a bit: Generate random $r \in Z_N^*$ and output $r^e \text{ mod } N$ and $\text{LSB}(r) \oplus m$.

- $Dec(sk, c)$: Recover r via RSA inversion.

<u>IND-secure under the RSA assumption</u>: given N, e (as above) and r^e mod N, hard to compute r.

Today:
Constructions of Public-Key Encryption

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2: Quadratic Residuosity/Goldwasser-Micali

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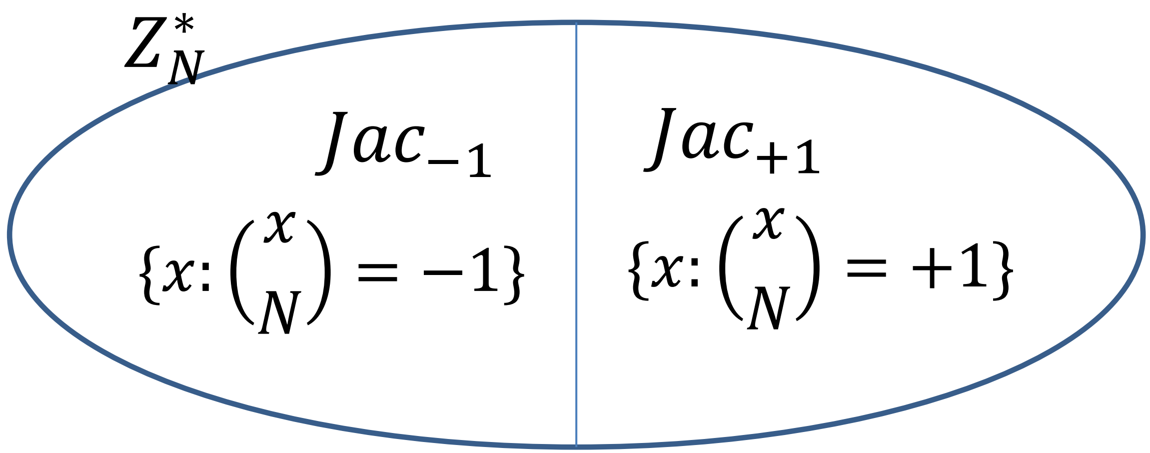
Quadratic Residuosity

Let's review some *more* number theory from L7-8.

Let $N = pq$ be a product of two large primes.

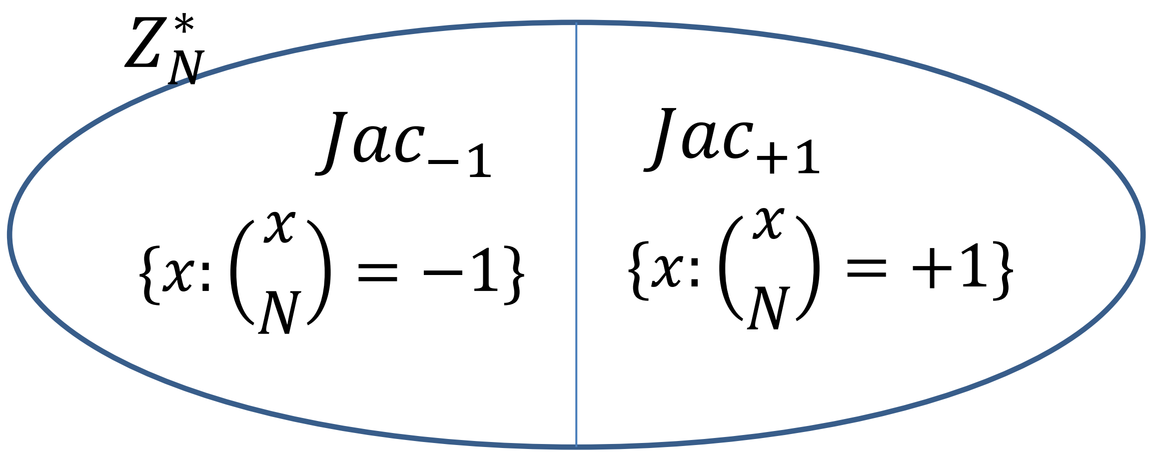


Jacobi symbol $\left(\frac{x}{N}\right) = \left(\frac{x}{p}\right)\left(\frac{x}{q}\right)$ is +1 if x is a square mod both p and q or a non-square mod both p and q.

Quadratic Residuosity

Let's review some *more* number theory from L7-8.

Let $N = pq$ be a product of two large primes.



Surprising fact: Jacobi symbol $\left(\frac{x}{N}\right) = \left(\frac{x}{p}\right)\left(\frac{x}{q}\right)$ is computable in poly time without knowing p and q.

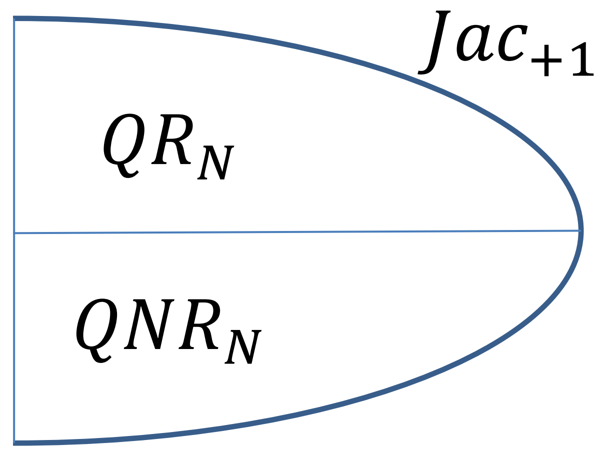
Quadratic Residuosity

Let's review some *more* number theory from L7-8.

Let $N = pq$ be a product of two large primes.

So: $QR_N = \{x: \left(\frac{x}{p}\right) = \left(\frac{x}{1}\right) = +1\}$

$QNR_N = \{x: \left(\frac{x}{p}\right) = \left(\frac{x}{1}\right) = -1\}$



QR_N is the set of squares mod N and QNR_N is the set of non-squares mod N with Jacobi symbol +1.

Quadratic Residuosity

Let's review some *more* number theory from L7-8.

Let $N = pq$ be a product of two large primes.

<u>Quadratic Residuosity Assumption (QRA)</u>

Let $N = pq$ be a product of two large primes.
No PPT algorithm can distinguish between a random element of QR_N from a random element of QNR_N given only N.

Goldwasser-Micali (GM) Encryption

$Gen(1^n)$: Generate random n-bit primes p and q and let $N = pq$. Let $y \in QNR_N$ be some quadratic non-residue with Jacobi symbol +1.

Let $pk = (N, y)$ and let $sk = (p, q)$.

$Enc(pk, b)$ where b is a bit:
Generate random $r \in Z_N^*$ and output $r^2 \bmod N$ if $b = 0$ and $r^2 y \bmod N$ if $b = 1$.

$Dec(sk, c)$: Check if c $\in Z_N^*$ is a quadratic residue using p and q. If yes, output 0 else 1.

Goldwasser-Micali (GM) Encryption

$Enc(pk, b)$ where b is a bit:
Generate random $r \in Z_N^*$ and output $r^2 \bmod N$ if $b = 0$ and $r^2 y \bmod N$ if $b = 1$.

IND-security follows directly from the quadratic residuosity assumption.

GM is a Homomorphic Encryption

Given a GM-ciphertext of b and a GM-ciphertext of b', I can compute a GM-ciphertext of $b + b' mod\ 2$.

without knowing anything about $\boldsymbol{b}$ or $\boldsymbol{b'}$!

$Enc(pk, b)$ where b is a bit:
Generate random $r \in Z_N^*$ and output $r^2 y^b \text{ mod } N$.

Claim: $Enc(pk, b) \cdot Enc(pk, b')$ is an encryption of $b \oplus b' = \ b + b' mod\ 2$.

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Constructions of Public-Key Encryption

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Diffie-Hellman Key Exchange

Commutativity in the exponent: $(g^x)^y = (g^y)^x$

(where g is an element of some group)

So, you can compute g^{xy} given either g^x and y, or g^y and x.

Diffie-Hellman Assumption (DHA):

Hard to compute g^{xy} given only g, g^x and g^y

Diffie-Hellman Key Exchange

Diffie-Hellman Assumption (DHA):

Hard to compute it given only g, g^x and g^y

We know that if discrete log is easy, DHA is false.

Major Open Problem:

Are discrete log and DHA equivalent?

Diffie-Hellman Key Exchange

p, g: Generator of our group Z_p^*

$g^x \bmod p$ →

← $g^y \bmod p$

Pick a random number $x \in Z_{p-1}$

Pick a random number y $\in Z_{p-1}$

Shared key K = $g^{xy} \bmod p$

= $(g^y)^x \bmod p$

Shared key K = $g^{xy} \bmod p$

= $(g^x)^y \bmod p$

Diffie-Hellman/El Gamal Encryption

- $Gen(1^n)$: Generate an n-bit prime p and a generator g of Z_p^*. Choose a random number $x \in Z_{p-1}$

  Let $pk = (p, g, g^x)$ and let $sk = x$.

- $Enc(pk, m)$ where $m \in Z_p^*$: Generate random $y \in Z_{p-1}$ and output $(g^y, g^{xy} \cdot m)$

- $Dec(sk = x, c)$: Compute g^{xy} using g^y and x and divide the second component to retrieve m.

Is this Secure?

The Problem

Claim: Given p, g, $g^x \bmod p$ and $g^y \bmod p$, adversary can compute some information about $g^{xy} \bmod p$.

determine if $g^{xy} \bmod p$ is a square mod p.

Corollary: Therefore, additionally given $g^{xy} \cdot m \bmod p$, the adversary can determine whether m is a square mod p, violating “IND-security”.

The Problem

Claim: Given p, g, $g^x \bmod p$ and $g^y \bmod p$, adversary can determine if $g^{xy} \bmod p$ is a square mod p.

$g^{xy} \bmod p$ is a square $\Leftrightarrow$ $xy \pmod{p-1}$ is even

$\Leftrightarrow$ xy is even

$\Leftrightarrow$ x is even or y is even

$\Leftrightarrow$ $x\ (mod\ p-1)$ is even or $y \pmod{\mathrm{p}-1}$ is even

$\Leftrightarrow$ $g^x\ mod\ p$ or $g^y\ mod\ p$ is a square

This can be checked in poly time!

Diffie-Hellman Encryption

Claim: Given p, g, $g^x \bmod p$ and $g^y \bmod p$, adversary can determine if $g^{xy} \bmod p$ is a square $\bmod p$.

More generally, dangerous to work with groups that have non-trivial subgroups (in our case, the subgroup of all squares mod p)

Lesson: Best to work over a group of prime order. Such groups have no subgroups.

An Example: Let $p = 2q + 1$ where q is a prime itself. Then, the group of squares mod p has order $\frac{(p-1)}{2} = q$.

Diffie-Hellman/El Gamal Encryption

- $Gen(1^n)$: Generate an n-bit "safe" prime $p = 2q + 1$ and a generator g of Z_p^* and let $h = g^2 \mod p$ be a generator of QR_p . Choose a random number $x \in Z_q$.

  Let $pk = (p, h, h^x)$ and let $sk = x$.

- $Enc(pk, m)$ where $m \in QR_p$: Generate random $y \in Z_q$ and output $(g^y, g^{xy} \cdot m)$

- $Dec(sk = x, c)$: Compute g^{xy} using g^y and x and divide the second component to retrieve m.

Decisional Diffie-Hellman Assumption

Decisional Diffie-Hellman Assumption (DDHA):

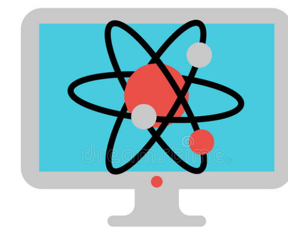
Hard to distinguish between g^{xy} and a uniformly random group element, given g, g^x and g^y

That is, the following two distributions are computationally indistinguishable:

$$(g, g^x, g^y, g^{xy}) \approx (g, g^x, g^y, u)$$

DH/El Gamal is IND-secure under the DDH assumption.

Today:
Constructions of Public-Key En



QUANTUM COMPUTER

1: ~~Trapdoor Permutations (RSA)~~

2: ~~Quadratic Residuosity/Goldwasser-Micali~~

3: ~~Diffie-Hellman/El Gamal~~

4: Learning with Errors/Regev

(post-quantum secure, as far as we know)

Solving Linear Equations

Solving *Noisy* Linear Equations

$$(\mathbf{s_1}|\mathbf{s_2})\begin{bmatrix}\mathbf{5} & \mathbf{1} & \mathbf{3}\\ \mathbf{6} & \mathbf{2} & \mathbf{1}\end{bmatrix} = [11 \quad 3 \quad 9]$$

Easy! ⇒ Find $(\mathbf{s_1}|\mathbf{s_2})$

How about:

$$(\mathbf{s_1}|\mathbf{s_2})\begin{bmatrix}\mathbf{5} & \mathbf{1} & \mathbf{3}\\ \mathbf{6} & \mathbf{2} & \mathbf{1}\end{bmatrix} + [e_1 \quad e_2 \quad e_3] = [11 \quad 3 \quad 9]$$

(e_1,e_2,e_3) are "small" numbers

Very hard! ⇒ Find $\vec{s}$

in large dimensions

Learning with Errors (LWE)

[Regev05, following BFKL93, Ale03]

very hard!

LWE: $(\mathbf{A}, s\mathbf{A}+e)$ ⇒ Find s

($\mathbf{A} \in Z_q^{nXm}$

$\mathbf{s} \in Z_q^n$ random "small" secret vector

$\boldsymbol{e} \in Z_q^n$: random "small" error vector)

Decisional LWE:

$(\mathbf{A}, \mathrm{s}\mathbf{A}+e) \approx_c (\mathbf{A}, \mathbf{b})$

(b uniformly random)

"Decisional LWE is as hard as LWE".

Basic (Secret-key) Encryption

[Regev05]

n = security parameter, q = "small" prime

- Secret key sk = Uniformly random vector $\mathbf{s} \in Z_q^n$
- Encryption $Enc_{\mathbf{s}}(m)$: // $m \in \{0,1\}$
  - Sample uniformly random $\mathbf{a} \in Z_q^n$, "short" noise $e \in Z$
  - The ciphertext $\mathbf{c} = (\mathbf{a}, b = \langle \mathbf{a}, \mathbf{s} \rangle + e + m \quad)$
- Decryption $Dec_{sk}(\mathbf{c})$: Output $(b - \langle \mathbf{a}, \mathbf{s} \rangle \bmod q)$

  // correctness as long as $|e| < q/4$

Basic (Secret-key) Encryption

[Regev05]

This is an incredibly cool scheme. In particular, additively homomorphic.

$\boldsymbol{c} = (\mathbf{a}, b = \langle \mathbf{a}, \mathbf{s} \rangle + e + m \lfloor q/2 \rfloor) \quad +$

$\boldsymbol{c}' = (\mathbf{a}', b' = \langle \mathbf{a}', \mathbf{s} \rangle + e' + m' \lfloor q/2 \rfloor)$

$\boldsymbol{c} + \boldsymbol{c}' = (\mathbf{a}+\mathbf{a}', b + b' = \langle \mathbf{a} + \mathbf{a}', \mathbf{s} \rangle + (e+e') + (m+m') \lfloor q/2 \rfloor)$

In words: $c + c'$ is an encryption of $m+m'$ (mod 2)

Public-key Encryption

[Regev05]

Here is a crazy idea. Public key has an encryption of 0 (call it c_0) and an encryption of 1 (call it c_1).
If you want to encrypt 0, output c_0 and if you want to encrypt 1, output c_1.

Well, turns out to be a crazy *bad* idea.

If only we could produce *fresh* encryptions of 0 or 1 given just the pk…

Public-key Encryption

[Regev05]

Here is another crazy idea.
Public key has *many* encryptions of 0 and an encryption of 1 (call it c_1).

If you want to encrypt 0, output a random linear combination of the 0-encryptions.

If you want to encrypt 1, output a random linear combination of the 0-encryptions plus c_1.

This one turns out to be a crazy ***good*** idea.

Public-key Encryption

[Regev05]

- Secret key sk = Uniformly random vector $\mathbf{s} \in Z_q^n$
- Public key pk: for $i\ from\ 1\ to\ k = poly(n)$

$$\left(\boldsymbol{c_0} = (\boldsymbol{a_0}, \langle \boldsymbol{a_0}, \boldsymbol{s} \rangle + e_0 + \left\lfloor \frac{q}{2} \right\rfloor), \boldsymbol{c_i} = (\boldsymbol{a_i}, \langle \boldsymbol{a_i}, \boldsymbol{s} \rangle + e_i)\right)$$

- Encrypting a bit m: pick k random bits $r_1, \ldots, r_k$

$$\sum_{i=1}^{k} r_i \boldsymbol{c_i} + m \cdot \boldsymbol{c_0}$$

Correctness: additive homomorphism

Security: decisional LWE + "Leftover Hash Lemma"

We saw:
Constructions of Public-Key Encryption

1: Trapdoor Permutations (RSA)

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Practical Considerations

I want to encrypt to Bob. How do I know his public key?

Public-key Infrastructure: a directory of identities together with their public keys.

<u>Needs to be "authenticated":</u>

otherwise Eve could replace Bob's pk with her own.

Practical Considerations

Public-key encryption is orders of magnitude slower than secret-key encryption.

1. We just showed how to encrypt bit-by-bit! Super-duper inefficient.
2. Exponentiation takes $O(n^2)$ time as opposed to typically linear time for secret key encryption (AES).
3. The n itself is large for PKE (RSA: $n \geq 2048$) compared to SKE (AES: $n = 128$).

Can solve problem 1 and minimize problems 2&3 using **hybrid encryption**.

Hybrid Encryption

To encrypt a long message m (think 1 GB):

Pick a random key K (think 128 bits) for a secret-key encryption

Encrypt K with the PKE: $PKE.Enc(pk, K)$

Encrypt m with the SKE: $\text{SKE}.Enc(K, m)$

To decrypt: recover K using sk. Then using K, recover m

Next Lecture:

Digital Signatures